1. Eigen decomposition II

To begin with, we prove the corollary 6 in the previous lecture notes.

Proof of corollary 6 in Lecture notes May 9th. A bounded sequence $\{u_i\}$ in H^1_0 contains a subsequence which is convergent strongly in L^2. Since the subsequence is also bounded, it contains its own subsequence which is convergent weakly in H^1_0. Namely, there exists a subsequence $\{u_{i_m}\}$ converges weakly to $\bar{u}_1$ in H^1_0 and converges strongly to $\bar{u}_2$ in L^2. Let us denote $u_{i_m} - \bar{u}_1$ by v_m and $\bar{u}_2 - \bar{u}_1$ by $\bar{v}$. Then, the bounded sequence $\{v_m\}$ converges weakly to 0 in H^1_0 and strongly to $\bar{v}$ in L^2.

Next, we recall a compactly supported rotationally symmetric smooth non-negative mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$ and $\eta(x) = 0$ for $|x| \geqslant \theta$ where $\theta \in (0,1)$, and the scaled mollifier $\eta_\delta(x) = \delta^{-n}(x/\delta)$. Notice that $\int_{\mathbb{R}^n} \eta_\delta(x)dx = 1$ and $\eta(x) = 0$ for $|x| \geqslant \delta\theta$. Then, given a function $f : \Omega \to \mathbb{R}$, we define the δ-mollified function $f_\delta \in C^\infty_c(\Omega_\delta)$ by

$$f_\delta(x) = \int_{\Omega_\delta} f(y)\eta_\delta(x-y)dy,$$

where $\Omega_r = \{x \in \mathbb{R}^n : |x-y| \leqslant r \text{ for some } y \in \Omega\}$. (We consider $f = 0$ in $\Omega_\delta \backslash \Omega$.) We observe that $\nabla(f_\delta) = (\nabla f)_\delta$ holds for any smooth function f. In addition, since η is rotationally symmetric, $f, g \in L^2(\Omega)$ satisfy

$$\int_{\mathbb{R}} f(x) g_\delta dx(x) = \int_{\Omega_\delta}\int_{\Omega_\delta} f(x)g(y)\eta_\delta(x-y)dydx = \int f_\delta(x)g(x)dx.$$

Now, we observe that $v_m \in H^1_0(\Omega_{2\delta})$. Hence, for any $\xi \in C^\infty_c(\Omega_{2\delta})$ we have

$$\langle v_m, \xi\rangle_{H^1(\Omega_{2\delta})} = \int_{\Omega_{2\delta}} \nabla v_m \nabla\xi + v_m\xi dx = \int_{\Omega_{2\delta}} v_m(-\Delta\xi + \xi)dx = \langle v_m, -\Delta\xi + \xi\rangle_{L^2(\Omega_{2\delta})}.$$

Passing $m \to \infty$ yields

$$0 = \langle \bar{v}, -\Delta\xi + \xi\rangle_{L^2(\Omega_{2\delta}}.$$

Now, we set $\xi = (\bar{v}_\delta)_\delta$. Then,

$$\begin{aligned} 0 &= \langle \bar{v}, -\Delta(\bar{v}_\delta)_\delta + (\bar{v}_\delta)_\delta\rangle_{L^2} = \langle \bar{v}, -(\Delta\bar{v}_\delta)_\delta + (\bar{v}_\delta)_\delta\rangle_{L^2} \\ &= \langle \bar{v}_\delta, -\Delta\bar{v}_\delta + \bar{v}_\delta\rangle_{L^2} = \int_{\Omega_{2\delta}} -\bar{v}_\delta\Delta\bar{v}_\delta + |\bar{v}_\delta|^2 dx \\ &= \int_{\Omega_{2\delta}} |\nabla\bar{v}_\delta|^2 + |\bar{v}_\delta|^2 dx. \end{aligned}$$

Namely, $v_\delta = 0$ in $\Omega_{2\delta}$. Passing $\delta \to 0$, we have $u = 0$ almost everywhere in L^2. □

Lemma 1. *Given $f \in H_0^1$, the following holds in L^2 sense.*

$$f = \sum_{i=1}^{\infty} \frac{\langle f, w_i \rangle_{L^2} w_i}{\|w_i\|_{L^2}^2}.$$

Proof. We define $a_i = \|w_i\|_{L^2}^{-2} \langle f, w_i \rangle_{L^2}$, and $f_k = f - \sum_{i=1}^k a_i w_i$. Then, for each $1 \leqslant j \leqslant k$, we have

$$\langle \nabla f_k, \nabla w_j \rangle_{L^2} = \langle \nabla f - \nabla \sum_{i=1}^k a_i w_i, \nabla w_j \rangle_{L^2} = \langle \nabla f, \nabla w_j \rangle_{L^2} - \sum_{i=1}^k a_i \langle \nabla w_i, \nabla w_j \rangle_{L^2}$$
$$= \lambda_j \langle f, w_j \rangle_{L^2} - \sum_{i=1}^k a_i \langle w_i, \nabla w_j \rangle_{L^2} = \lambda_j a_j \|w_j\|_{L^2}^2 - a_j \|\nabla w_j\|_{L^2}^2 = 0.$$

Moreover,

$$\langle f_k, w_j \rangle_{L^2} = \langle f - \sum_{i=1}^k a_i w_i, w_j \rangle_{L^2} = \langle f, w_j \rangle_{L^2} - \sum_{i=1}^k a_i \langle w_i, w_j \rangle_{L^2} = a_j \|w_j\|_{L^2}^2 - a_j \|w_j\|_{L^2}^2 = 0.$$

Hence, we have $f_k \in X_k = \operatorname{span}(w_1, \cdots, w_k)^\perp$, and thus

$$\frac{\int |\nabla f_k|^2 dx}{\int |f_k|^2 dx} \geqslant \inf_{u \in X_k} \frac{\int |\nabla u|^2 dx}{\int |u|^2 dx} = \lambda_{k+1}.$$

However,

$$\|\nabla f\|_{L^2}^2 = \|\nabla f_k + \sum_{i=1}^k a_i w_i\|_{L^2}^2 = \|\nabla f_k\|_{L^2} + \sum_{i=1} \|a_i \nabla w_i\|_{L^2}^2 \geqslant \lambda_{k+1} \|f_k\|_{L^2}.$$

Therefore, $\lim \lambda_n = +\infty$ implies that passing $k \to \infty$ yields $f_k \to 0$ in L^2. □

Theorem 2. *Given $f \in L^2$, the following holds in L^2 sense.*

$$f = \sum_{i=1}^{\infty} \frac{\langle f, w_i \rangle_{L^2} w_i}{\|w_i\|_{L^2}^2}.$$

Proof. We recall that H_0^1 is dense in L^2. Hence, given $f \in L^2$, there exists a sequence $\{f^k\} \subset H_0^1$ which is convergent strongly to f in L^2 sense. Namely, given $\epsilon > 0$ there exists some large N such that $\|f - f^N\|_{L^2} \leqslant \frac{\epsilon}{3}$. Since $f_N \in H_0^1$, the previous lemma show that for sufficiently large M

$$\|f^N - \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f^N, w_i \rangle w_i\|_{L^2} \leqslant \tfrac{\epsilon}{3}$$

holds. Therefore,

$$\|f - \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f^N, w_i \rangle w_i\|_{L^2} \leqslant \tfrac{2}{3}\epsilon.$$

Moreover,

$$\| \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f, w_i \rangle w_i - \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f^N, w_i \rangle w_i \|_{L^2}^2 = \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f - f^N, w_i \rangle_{L^2}^2 \leqslant \|f - f^N\|_{L^2}^2 \leqslant \frac{\epsilon^2}{9}.$$

In conclusion,

$$\|f - \sum_{i=1}^{M} \|w_i\|_{L^2}^{-2} \langle f, w_i \rangle w_i\|_{L^2} \leqslant \epsilon.$$

This completes the proof. □

2. Applications

A direct application of the spectral analysis (the eigen decomposition) is the following theorem.

Theorem 3. *Given $f \in L^2(\Omega)$, the Dirichlet problem, $\Delta u = f$ in Ω and $u = 0$ in $\partial\Omega$, has a weak solution $u \in H_0^1(\Omega)$*

$$u = -\sum_{i=1}^{\infty} \frac{\langle f, w_i \rangle w_i}{\lambda_i \|w_i\|_{L^2}^2}.$$

Indeed, the weak solution would be good enough thank to the following regularity theory. (See Evans' PDE Ch 6.3 for details.) The following regularity also guarantees the smoothness of egien-functions.

Theorem 4. *Suppose that $a_{ij}, b_i, c, f \in C^\infty(\overline{\Omega})$, and Ω is smooth. Also, a_{ij} is uniformly elliptic, namely there exists some constant $0 < \lambda \leqslant \Lambda$ such that $\lambda\|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2$ holds for all $\xi \in \mathbb{R}^2$. Then, a weak solution $u \in H_0^1(\Omega)$ to the Dirichlet problem, $u = 0$ on $\partial\Omega$, to the equation*

$$f = \nabla_i(a_{ij}\nabla_j u) + b_i \nabla_i u + cu, \tag{1}$$

is of class $C^\infty(\overline{\Omega})$.

Moreover, we can obtain the following result.

Theorem 5 (Fredholm alternative)**.** *Given $f \in L^2(\Omega)$, the Dirichlet problem, $\Delta u + cu = f$ in Ω with constant c and $u = 0$ in $\partial\Omega$, has*

(i) *a unique solution if c is not an eigenvalue,*

(ii) *no solution if c is an eigenvalue and for some eigenfunction w of $\Delta w + cw = 0$ we have $\langle f, w\rangle_{L^2} \neq 0$,*

(iii) *infinitely many solution if c is an eigenvalue and $\langle f, w\rangle_{L^2} = 0$ holds for every eigenfunction w of $\Delta w + cw = 0$.*

3. The first eigenfunction

We observe an interesting property of the first eigenfunction.

Theorem 6 (Courant nodal domain)**.** *Let Ω be bounded and smooth. Then, an eigenfunction w of the first eigenvalue λ satisfies $w \neq 0$ in Ω.*

Proof. We recall that $w^+ = w$ if $w > 0$ and $w^+ = 0$ is $w \leqslant 0$, and $w^- = w^+ - w$. Then, $w^+ \in H_0^1$ and $\nabla w^+ = 0$ where $\{w \leqslant 0\}$. In the same manner, $w^- \in H_0^1$ and $\nabla w^- = 0$ in $w > 0$.

Without loss of generality, we assume $\|w\|_{L^2} = 1$. Then,

$$\lambda = \|\nabla w\|_{L^2}^2 = \|\nabla w^+ + \nabla w^-\|_{L^2}^2 = \|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2 \geqslant \lambda\|w^+\|_{L^2}^2 + \lambda\|w^-\|_{L^2}^2 = \lambda.$$

Therefore, $w^\pm$ are also minimizers of the ratio $\|\nabla u\|_{L^2}^2/\|u\|_{L^2}^2$ in H_0^1. Therefore, they are also eigenfunctions. Then, the regularity theory shows that $w^\pm \in C^\infty(\overline{\Omega})$.

Now, we observe that $\lambda > 0$ implies

$$\Delta w^\pm = -\lambda w^\pm \leqslant 0,$$

in Ω, namely $w^\pm$ are superharmonic. Therefore, if $w^\pm(x_\pm) = 0$ at some point $w_\pm \in \Omega$ then $w^\pm = 0$ by the strong maximum principle. Hence, we have $w^+ = 0$ or $w^- = 0$. Without loss of generality, we may assume $w^- = 0$. Then, w is again superharmonic. Hence, $w > 0$ in Ω. ◻

Indeed, we can consider eigenfunctions for more general operators. Then, the Courant nodal domain also works. Please, see the practice problem set. We state the Harnack's inequality for general linear elliptic equation in what follows. See Evans section 6.4 Theorem 5 for the proof.

Theorem 7 (Harnack)**.** *Suppose that* $a_{ij}, b_i, c \in C^\infty(\overline{\Omega})$*, and* Ω *is smooth. Also,* a_{ij} *is uniformly elliptic, namely there exists some constant* $0 < \lambda \leqslant \Lambda$ *such that* $\lambda\|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2$ *holds for all* $\xi \in \mathbb{R}^2$*. Suppose that a solution* $u \in C^2(\Omega)$ *to the elliptic equation* $\nabla_i(a_{ij}\nabla_j u) + b_i\nabla_i u + cu = 0$ *is non-negative,* $u \geqslant 0$ *in* Ω*. Then, for each compact set* K *in* Ω*, there exists some constant* C *such that*

$$\sup_K u \leqslant C \inf_K u.$$